Problem 1: Conservative vector fields

The work done by a force to move an object from point P0 to P1 is defined as

$$W = \int_{P0}^{P1} \overrightarrow{F} \cdot \overrightarrow{dl}$$

a **If the work done by the electric field to move a charge q between two points is independent of the path that the charge takes, show that:**

$$\oint_C \overrightarrow{E} \cdot \overrightarrow{dl} = 0$$

where C can be any closed path. Use the Stokes theorem to show that this is equivalent to $\nabla \times \overrightarrow{E} = 0$

If the work done to move a charge is not dependent on path length, we can say that the work of moving the charge from point P0 to P1 and back again from point P1 to P0 is the same as the work to keep the charge at point P0:

$$\int_{P0}^{P1} \overrightarrow{F} \cdot \overrightarrow{dl} + \int_{P1}^{P0} \overrightarrow{F} \cdot \overrightarrow{dl} = \int_{P0}^{P0} \overrightarrow{F} \cdot \overrightarrow{dl}$$

where $\int_{P0}^{P0} \overrightarrow{F} \cdot \overrightarrow{dl} = 0$ So,

$$\int_{P0}^{P1} \overrightarrow{F} \cdot \overrightarrow{dl} + \int_{P1}^{P0} \overrightarrow{F} \cdot \overrightarrow{dl} = 0$$

Moving the charge from point P0 to P1 and back to P0 is the closed integral C

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{dl} = 0$$

recall that $\overrightarrow{F} = q\overrightarrow{E}$, so

$$\oint_C q\overrightarrow{E} \cdot \overrightarrow{dl} = 0$$
$$\oint_C \overrightarrow{E} \cdot \overrightarrow{dl} = 0$$

From Stokes' Theorem, we have

$$\int_S (\nabla \times \overrightarrow{E}) \cdot dS = \oint_C \overrightarrow{E} \cdot \overrightarrow{dl}$$

which we can equate to 0 given our work above:

$$\int_S (\nabla \times \overrightarrow{E}) \cdot dS = \oint_C \overrightarrow{E} \cdot \overrightarrow{dl} = 0$$
$$\int_S (\nabla \times \overrightarrow{E}) \cdot dS = 0$$
$$\nabla \times \overrightarrow{E} = 0$$

b **Find *a, b, c* such that $\overrightarrow{E} = (x + 2y + az)\hat{x} + (bx - 3y - z)\hat{y} + (4x + cy + 2z)\hat{z}$ is a conservative field, i.e. $\nabla \times \overrightarrow{E} = 0$**

Because we are given the vector for the electric field, we can simply find its curl and then later plug in values for the constants to make it equal to 0.

$$\nabla \times \overrightarrow{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= \hat{x}\left(\frac{\partial(4x + cy + 2z)}{\partial y} - \frac{\partial(bx - 3y - z)}{\partial z}\right) + \hat{y}\left(\frac{\partial(x + 2y + az)}{\partial z} - \frac{\partial(4x + cy + 2z)}{\partial x}\right)$$

$$+\hat{z}\left(\frac{\partial(bx - 3y - z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial y}\right)$$

$$\nabla \times \overrightarrow{E} = \hat{x}(c + 1) + \hat{y}(a - 4) + \hat{z}(b - 2)$$

To obtain $\nabla \times \overrightarrow{E} = 0$, then clearly c = -1, a = 4, and b = 2.

c **Find its scalar potential *V(x,y,z)* such that: $\overrightarrow{E} = -\nabla V$**

To find the potential, we must integrate each component as follows:

$$V_x = -\int E_x dx = -\int (x + 2y + 4z)dx = -\frac{x^2}{2} - 2xy - 4xz$$

$$V_y = -\int E_y dy = -\int (2x - 3y - z)dy = -2xy + \frac{3}{2}y^2 + yz$$

$$V_z = -\int E_z dz = -\int (4x - y + 2z)dz = -4xz + yz - z^2$$

The easy way of bringing the above together to find V(x,y,z) is by adding the unique terms from each of the above expressions and a constant of integration:

$$V(x, y, z) = -\frac{x^2}{2} - 2xy - 4xz + \frac{3}{2}y^2 + yz - z^2 + C$$

d **Obtain $\nabla \cdot \overrightarrow{E}$**

$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1}$$

substitute components of $\overrightarrow{E}$ and constants found in part b:

$$\nabla \cdot E = \frac{\partial}{\partial x}(x + 2y + 4z) + \frac{\partial}{\partial y}(2x - 3y - z) + \frac{\partial}{\partial z}(4x - y + 2z)$$

$$\nabla \cdot E = 1 - 3 + 2$$

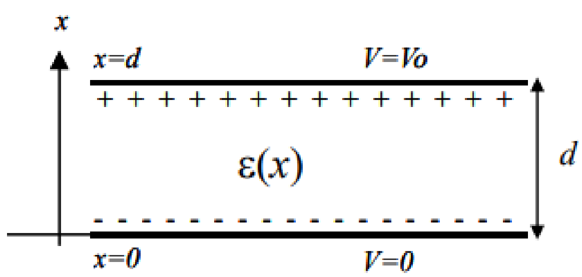
$$\nabla \cdot E = 0$$

Problem 2: Parallel Plate Capacitor

Consider the parallel plate capacitor shown in the figure below. The region between the plates is filled with a perfect dielectric of non-uniform permittivity, $\epsilon(x)$:

$$\epsilon(x) = \frac{\epsilon_0}{1 - \frac{x}{2d}} \tag{2}$$

The electric potential, $V(x)$, at $x = d$, is V_0, and its reference is taken at $x = 0$, such that $V(x = 0) = 0$.



a **Obtain the electric potential, $V(x)$, for the region between the plates ($0 < x < d$).**

Beginning with Maxwell's Equations, we have

$$\nabla \cdot \overrightarrow{D} = \rho_\nu(x) = 0 \quad \text{because there is no free charge}$$

This problem is one dimensional, so we can write

$$\nabla \cdot \overrightarrow{D} \Longrightarrow \frac{\partial}{\partial x} D_x = 0 \Longrightarrow D_x(x) = A$$

where A is a constant.

Recall the constitutive relation:

$$\overrightarrow{D} = \epsilon \overrightarrow{E} \Longrightarrow \overrightarrow{E} = \frac{1}{\epsilon(x)}\overrightarrow{D} = \frac{A}{\epsilon(x)}\hat{x}$$

$$\overrightarrow{E}(x) = \frac{A}{\epsilon_0}\left(1 - \frac{x}{2d}\right)\hat{x}$$

Now we can relate $\overrightarrow{E}$ and V through the relation

$$\overrightarrow{E} = -\nabla V = -\frac{\partial V}{\partial x} = \frac{A}{\epsilon_0}\left(1 - \frac{x}{2d}\right)\hat{x}$$

$$\int dV = -\int \frac{A}{\epsilon_0}\left(1 - \frac{x}{2d}\right)dx$$

$$V(x) = -\frac{A}{\epsilon_0}\left(x - \frac{x^2}{4d}\right) + B \quad \text{(where B is a constant from the integration)}$$

Now apply our boundary conditions, $V(x=0) = 0$ and $V(x=d) = V_0$ to find A and B.

$$V(x=0) = 0 = B \Longrightarrow B = 0$$
$$V(x=d) = -\frac{A}{\epsilon_0}\left(d - \frac{d}{4}\right) = V_0 \Longrightarrow A = -\frac{4}{3d}\epsilon_0 V_0$$

So

$$V(x) = \frac{4}{3d}V_0\left(x - \frac{x^2}{4d}\right)$$

b **Obtain an expression for the capacitance per unit area.**

$$\overrightarrow{E} = \frac{A}{\epsilon_0}\left(1 - \frac{x}{2d}\right)\hat{x} = -\frac{4}{3d}V_0\left(1 - \frac{x}{2d}\right)\hat{x}$$

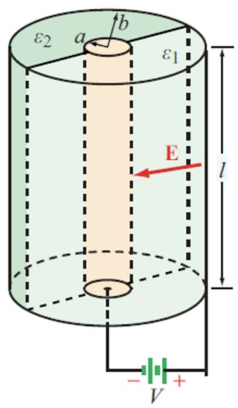
$$\overrightarrow{D} = \epsilon(x)\overrightarrow{E} = -\frac{4}{3d}V_0\epsilon_0\hat{x}$$

$$\rho_s = \overrightarrow{D} \cdot \hat{n} = -\hat{x} \cdot \overrightarrow{D} = \frac{4}{3d}V_0\epsilon_0$$

$$C_s \equiv \frac{\text{Capacitance}}{\text{unit area}} = \frac{\rho_s}{V_0} = \frac{4}{3d}\epsilon_0$$

Problem 3: Coaxial Capacitor

A coaxial capacitor consists of two concentric, conducting, cylindrical surfaces, one of radius a and another of radius b, as shown in the figure below. The insulating layer separating the two conducting surfaces is divided equally into two semi-cylindrical sections, one filled with dielectric ϵ_1 and the other filled with dielectric ϵ_2.



a **Develop an expression for the coax capacitance, C, in terms of length, l, the geometrical parameters, and the dielectric constants.**

This coaxial capacitor can be viewed as 2 different capacitors in parallel with one another. Because capacitors in parallel with one another have their capacitance simply add, we can solve for each half independently and sum our solutions at the end for the final expression.

For the half with dielectric ϵ_1, for given voltage V across the capacitor, charges $+Q_1$ and $-Q_1$ accumulate on the surfaces of the outer and inner conductors respectively. We assume that the charges are uniformly distributed along the length of the conductors with surface charge density $\rho'_{s1} = \frac{Q_1}{\pi bl}$ on the outer conductor and $\rho_{s1}" = -\frac{Q_1}{\pi al}$ on the inner one.

We can construct a cylinder Gaussian surface in the dielectric in between the conductors, with radius r such that $a < r < b$. From Gauss's law

$$\overrightarrow{E} = -\hat{r}\frac{Q_1}{\pi\epsilon_1 rl}$$

The potential difference between the outer and inner conductor is

$$V_1 = -\int_a^b \overrightarrow{E} \cdot \overrightarrow{dl} = -\int_a^b (-\hat{r}\frac{Q_1}{\pi\epsilon_1 rl}) \cdot (\hat{r}dr) = \frac{Q_1}{\pi\epsilon_1 l} \ln\left(\frac{b}{a}\right)$$

Similarly

$$V_2 = \frac{Q_2}{\pi\epsilon_2 l} \ln\left(\frac{b}{a}\right)$$

We know that $V_1 = V_2$. This also lets us know that $Q_1 = \frac{\epsilon_1}{\epsilon_2} Q_2$, though you won't need to use that in this problem.

The capacitance C is given by

$$C_1 = \frac{Q_1}{V} = \frac{\pi \epsilon_1 l}{\ln(b/a)}$$
$$C_2 = \frac{Q_2}{V} = \frac{\pi \epsilon_2 l}{\ln(b/a)}$$

Giving us a combined C of

$$C = C_1 + C_2 = \frac{\pi l}{\ln(b/a)}(\epsilon_1 + \epsilon_2)$$

This makes intuitive sense, as the equation for the capacitance of concentric cylindrical shells separated by a single dielectric ϵ is $C = \frac{2\pi\epsilon l}{\ln(b/a)}$. This matches our equation, with ϵ being the average between the two dielectrics ϵ_1 and ϵ_2.

b **Evaluate the value of C for $a = 6$ mm, $b = 10$ mm, $\epsilon_{1r} = 2$, $\epsilon_{2r} = 6$, and $l = 8$ cm.**

First note that

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

Therefore,

$$\epsilon_1 = \epsilon_{1r}\epsilon_0 = 2\epsilon_0$$
$$\epsilon_2 = \epsilon_{2r}\epsilon_0 = 6\epsilon_0$$

Simply plug into the expression we derived ($\epsilon_0 \approx 8.854 \cdot 10^{-12}$)

$$C = \frac{\pi \cdot 8 \cdot 10^{-2}}{\ln(10/6)}(2+6) \cdot 8.854 \cdot 10^{-12} = 3.485 \cdot 10^{-11}\ (F) = 34.85\ (pF)$$